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Iterations of the non-classical symmetries method and conditional Lie–Bäcklund symmetries

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Abstract. Iterations of the non-classical symmetries method give rise to new nonlinear equations, which inherit the Lie point symmetry algebra of the original equation. Invariant solutions of these heir equations supply new solutions of the original equation. We show that particular cases of such invariant solutions correspond to Zhdanov's conditional Lie–Bäcklund symmetries.

1. Introduction

The most famous and established method for finding exact solutions of differential equations is the classical symmetries method (CSM), also called group analysis, which originated in 1881 from the pioneering work of Sophus Lie [1]. Many good books have been dedicated to this subject and its generalizations [2–9].

The nonclassical symmetries method (NSM) was introduced in 1969 by Bluman and Cole [10] to obtain new exact solutions of the linear heat equation, i.e. solutions not deducible from the classical symmetries method. The NSM consists in adding the invariant surface condition to the given equation, and then applying the CSM. The main difficulty of this approach is that the determining equations are no longer linear. On the other hand, the NSM may give more solutions than the CSM. The NSM has been successfully applied to various equations [11–16]‡, for the purpose of finding new exact solutions.

Recently, Galaktionov [17] and King [18] have found exact solutions of certain evolution equations which apparently do not seem to be derived by either the CSM or NSM. In [19], we have shown how these solutions can be obtained by iterating the NSM. A special case of the NSM generates a new nonlinear equation (the so-called G-equation [20]), which inherits the prolonged symmetry algebra of the original equation. Another special case of the NSM is then applied to this heir-equation to generate another heir-equation, and so on. Invariant solutions of these heir-equations are just the solutions derived in [17, 18].

In this paper, we show that invariant solutions of the heir-equations also yield Zhdanov's conditional Lie–Bäcklund symmetries [21].

The use of a symbolic manipulator becomes imperative, because the heir-equations can be quite long: one more independent variable is added at each iteration. We employ our own interactive REDUCE programs [22] to calculate both the classical and the non-classical symmetries, and generate the heir-equations.

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‡ Just to cite some of the numerous papers on this subject.

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2. Iterating the non-classical symmetries method

Let us consider an evolution equation in two independent variables and one dependent variable†:

$$u_t = H(t, x, u, u_x, u_{xx}, u_{xxx}, \ldots). \tag{1}$$

The invariant surface condition is given by:

$$V_1(t, x, u)u_t + V_2(t, x, u)u_x = G(t, x, u). \tag{2}$$

Let us take the case with $V_1 = 0$ and $V_2 = 1$, so that (2) becomes

$$u_x = G(t, x, u). \tag{3}$$

Applying the NSM leads to an equation for G. We call this equation the G-equation [20]. Its invariant surface condition is given by

$$\xi_1(t, x, u, G)G_t + \xi_2(t, x, u, G)G_x + \xi_3(t, x, u, G)G_u = \eta(t, x, u, G). \tag{4}$$

Let us consider the case $\xi_1 = 0$, $\xi_2 = 1$, and $\xi_3 = G$, so that (4) becomes

$$G_x + GG_u = \eta(t, x, u, G). \tag{5}$$

Applying the NSM leads to an equation for η. We call this equation the η-equation. Clearly

$$G_x + GG_u \equiv u_{xx} \equiv \eta. \tag{6}$$

We could keep iterating to obtain the Ω-equation, which corresponds to

$$\eta_x + G\eta_u + \eta\eta_G \equiv u_{xxx} \equiv \Omega(t, x, u, G, \eta) \tag{7}$$

the ρ-equation, which corresponds to

$$\Omega_x + G\Omega_u + \eta\Omega_G + \Omega\Omega_\eta \equiv u_{xxxx} \equiv \rho(t, x, u, G, \eta, \Omega) \tag{8}$$

and so on. Each of these equations inherits the symmetry algebra of the original equation, with the right prolongation: first prolongation for the G-equation, second prolongation for the η- equation, and so on.

This iterating method yields both partial symmetries as given by Vorob'ev in [24], and differential constraints as given by Olver [25]. Also, it should be noticed that the $u_{\underbrace{xx\ldots}_{n}}$– equation of (1) is just one of many possible n-extended equations as defined by Guthrie in [26].

More details can be found in [19].

3. Zhdanov's conditional Lie–Bäcklund symmetries

In [21], Zhdanov introduced the concept of conditional Lie–Bäckund symmetry, i.e. given an evolution-type equation (1) and some smooth Lie–Bäcklund vector field (LBVF)

$$Q = S\partial_u + (D_t S)\partial_{u_t} + (D_x S)\partial_{u_x} + \cdots \tag{9}$$

with

$$S = S(t, x, u, u_t, u_x, \ldots)$$

then equation (1) is said to be conditionally invariant under LBVF (9) if the condition

$$Q(u_t - H)|_{M\cap L_x} = 0 \tag{10}$$

† In [23] it was shown how to iterate the NSM in the case of systems.

holds, with M a set of all differential consequences of the equation (1), and L_x a set of all x-differential consequences of the equation $S = 0$. Zhdanov claimed that this definition can be applied to construct new exact solutions of (1), which cannot be obtained by either Lie point or Lie–Bäcklund symmetries.

Instead, $S = 0$ is just a particular invariant solution of a suitable heir-equation generated by iterating the NSM. Of course, we assume that $S = 0$ can be written in explicit form with respect to the highest derivative of u.

As *example 1*, Zhdanov considered the following nonlinear heat conductivity equation with a logarithmic-type nonlinearity

$$u_t = u_{xx} + (\alpha + \beta \log(u) - \gamma^2 \log(u)^2)u \tag{11}$$

and obtained new solutions by showing that (11) is conditionally invariant with respect to LBVF (9) with

$$S = u_{xx} - \gamma u_x - u_x^2/u. \tag{12}$$

It can be easily shown that equation

$$S \equiv u_{xx} - \gamma u_x - u_x^2/u = 0 \tag{13}$$

admits an eight-dimensional Lie point symmetry algebra and therefore is linearizable†. In fact, the change of dependent variable $u = \exp(v)$ transforms (13) into $v_{xx} - \gamma v_x = 0$, which can be easily integrated. Therefore, the following general solution of (13) can be obtained [21],

$$u(t, x) = \exp\left(\phi_1(t) + \phi_2(t)\exp(\gamma x)\right)$$

which substituted into (11) gives rise to the following system of two ordinary differential equations,

$$\dot{\phi}_1 = \alpha + \beta\phi_1 - \gamma^2\phi_1^2 \qquad \dot{\phi}_2 = (\beta + \gamma^2 - 2\gamma^2\phi_1)\phi_2$$

and its general solution can easily be derived [21].

Now, let us apply the iterations of the NSM to equation (11). Its G-equation is

$$\begin{aligned}2G_{xu}G + G_{uu}G^2 + G_u \log(u)^2\gamma^2 u - G_u \log(u)\beta u - G_u\alpha u - G_t \\ +G_{xx} - \log(u)^2 G\gamma^2 + \log(u)\beta G - 2\log(u)G\gamma^2 + \alpha G + \beta G = 0.\end{aligned} \tag{14}$$

Its η-equation is

$$\begin{aligned}2\eta_{uG}\eta G u + \eta_{GG}\eta^2 u + \eta_G \log(u)^2\gamma^2 G u - \eta_G \log(u)\beta G u + 2\eta_G \log(u)\gamma^2 G u \\ -\eta_G\alpha G u - \eta_G\beta G u + \eta_{uu}G^2 u + \eta_u \log(u)^2\gamma^2 u^2 - \eta_u \log(u)\beta u^2 \\ -\eta_u\alpha u^2 - \log(u)^2\gamma^2\eta u + \log(u)\beta\eta u - 2\log(u)\gamma^2\eta u - 2\log(u)\gamma^2 G^2 \\ +\alpha\eta u + \beta\eta u + \beta G^2 - 2\gamma^2 G^2 = 0.\end{aligned} \tag{15}$$

The Lie point symmetry algebra of (11) is spanned by the two vector fields $X_1 = \partial_t$, and $X_2 = \partial_x$. Therefore, (x, t)-independent invariant solutions of (15) are given in the form $\eta = \eta(u, G)$. A particular case is $\eta = r_1(u)G^2 + r_2(u)G + r_3(u)$, i.e. a polynomial of second degree in G. Substituting into (15) and assuming $r_3 = 0$ gives rise to

$$\eta = \frac{G^2}{u} \pm \gamma G. \tag{16}$$

Finally, substituting $\eta = u_{xx}$, and $G = u_x$ into (16) yields (13).

† Zhdanov integrated equation (13) without any mention of this property.

As *example 2*, Zhdanov considered the following nonlinear heat conductivity equation,

$$u_t = u_{xx} + F(u) \tag{17}$$

and established that it is conditionally invariant with respect to LBVF (9) with

$$S = u_{xx} - A(u)u_x^2 \tag{18}$$

if $F(u)$ and $A(u)$ satisfy the following system:

$$A'' + 4AA' + 2A^3 = 0 \tag{19}$$

$$F'' - A'F - AF' = 0. \tag{20}$$

Let us apply the iterations of the NSM to equation (17). Its G-equation is

$$F'(u)G + 2G_{xu}G + G_{uu}G^2 - FG_u - G_t + G_{xx} = 0. \tag{21}$$

Its η-equation is

$$F''G^2 - F'G\eta_G + F'\eta + 2\eta\eta_{xG} + 2G\eta\eta_{uG} + \eta^2\eta_{GG} - \eta_t + 2G\eta_{xu} + \eta_{xx} + G^2\eta_{uu} - F\eta_u = 0. \tag{22}$$

The Lie point symmetry algebra of (17) (with F an arbitrary function of u) is spanned by the two vector fields $X_1 = \partial_t$, and $X_2 = \partial_x$. Therefore, (x, t)-independent invariant solutions of (22) are given in the form $\eta = \eta(u, G)$. A particular case is $\eta = A(u)G^2 + B(u)G + C(u)$, i.e. a polynomial of second degree in G. Substituting into (22) yields

$$A'' + 4AA' + 2A^3 = 0 \tag{23}$$

$$4A'C - A'F + 2B'B + C'' + F'' - F'A + 4A^2C + 2AB^2 = 0 \tag{24}$$

$$2B'C - B'F + 4ABC = 0 \tag{25}$$

$$-C'F + F'C + 2AC^2 = 0. \tag{26}$$

If we assume $B = C = 0$, then $\eta = A(u)G^2$ and system (23)–(26) reduces to system (19) and (20).

As *example 3*, Zhdanov considered the following nonlinear equation

$$u_t = u_{xx} + a\log(u)^2 u \qquad (a \in \mathbb{R}^1) \tag{27}$$

and established that it is conditionally invariant with respect to LBVF (9) with

$$S = u^2u_{xxx} - 3uu_xu_{xx} + 2u_x^3 + au_xu^2. \tag{28}$$

Let us apply the iterations of the NSM to equation (27). Its G-equation is

$$2GG_{xu} + G^2G_{uu} - G_u\log(u)^2au - G_t + G_{xx} + \log(u)^2aG + 2\log(u)aG = 0. \tag{29}$$

Its η-equation is

$$2\eta_{xG}\eta u + 2\eta uG\eta Gu + \eta_{GG}\eta^2u - \eta_G\log(u)^2aGu - 2\eta_G\log(u)aGu - \eta_tu + 2\eta_{xu}Gu + \eta_{xx}u + \eta_{uu}G^2u - \eta_u\log(u)^2au^2 + \log(u)^2a\eta u + 2\log(u)a\eta u + 2\log(u)aG^2 + 2aG^2 = 0. \tag{30}$$

Its Ω-equation is

$$2\Omega_{x\eta}\Omega u^2 + 2\Omega_{u\eta}\Omega u^2G + 2\Omega_{\eta G}\Omega\eta u^2 + \Omega_{\eta\eta}\Omega^2u^2 - \Omega_\eta\log(u)^2a\eta u^2 - 2\Omega_\eta\log(u)a\eta u^2 - 2\Omega_\eta\log(u)auG^2 - 2\Omega_\eta auG^2 - \Omega_tu^2 + 2\Omega_{xu}u^2G + 2\Omega_{xG}\eta u^2 + \Omega_{xx}u^2 + 2\Omega_{uG}\eta u^2G + \Omega_{uu}u^2G^2 - \Omega_u\log(u)^2au^3 + \Omega_{GG}\eta^2u^2 - \Omega_G\log(u)^2au^2G - 2\Omega_G\log(u)au^2G + \log(u)^2a\Omega u^2 + 2\log(u)a\Omega u^2 + 6\log(u)a\eta uG - 2\log(u)aG^3 + 6a\eta uG = 0. \tag{31}$$

The Lie point symmetry algebra of (27) is spanned by the two vector fields $X_1 = \partial_t$, and $X_2 = \partial_x$. Therefore, (x, t)-independent invariant solutions of (31) are given in the form $\Omega = \Omega(u, G, \eta)$. A particular case is $\Omega = R_1(u, G)\eta + R_2(u, G)$, i.e. a polynomial of first degree in η. Substituting into (31) yields

$$\Omega = (3uG\eta - 2G^3 - au^2G)/u^2. \tag{32}$$

Finally, substituting $\Omega = u_{xxx}$, $\eta = u_{xx}$, and $G = u_x$ into (32) yields (28).

As *example 4*, Zhdanov considered all PDEs of the form

$$u_t = u_{xx} + R(u, u_x) \tag{33}$$

and established that they are conditionally invariant with respect to LBVF (9) with

$$S = u_{xx} - au \qquad (a \in \mathbb{R}^1) \tag{34}$$

if R satisfies the following equation†,

$$a^2u^2R_{u_xu_x} + 2auu_xR_{uu_x} + u_x^2R_{uu} + auR_u + au_xR_{u_x} - aR = 0 \tag{35}$$

i.e.

$$R = f_1(u_x^2 - au^2)u_x + f_2(u_x^2 - au^2)u. \tag{36}$$

Let us apply the iterations of the NSM to PDEs (33). Their G-equation is ($G \equiv u_x$)

$$R_GG_uG + R_GG_x + R_uG + 2GG_{xu} + G^2G_{uu} - RG_u - G_t + G_{xx} = 0. \tag{37}$$

Their η-equation is

$$\begin{aligned}2R_{uG}\eta G + R_{GG}\eta^2 + R_G\eta_x + R_G\eta_uG + R_{uu}G^2 - R_u\eta_GG + R_u\eta + 2\eta_{xG}\eta \\ +2\eta_{uG}\eta G + \eta_{GG}\eta^2 - \eta_t + 2\eta_{xu}G + \eta_{xx} + \eta_{uu}G^2 - \eta_uR = 0.\end{aligned} \tag{38}$$

The Lie point symmetry algebra of (33) (with R an arbitrary function of u and u_x) is spanned by the two vector fields $X_1 = \partial_t$, and $X_2 = \partial_x$. Therefore, (x, t)-independent invariant solutions of (38) are given in the form $\eta = \eta(u, G)$. A particular case is $\eta = A(u)$, i.e. η independent by G. Substituting into (38) yields

$$2R_{uG}AG + R_{GG}A^2 + R_GA'G + R_{uu}G^2 + R_uA + A''G^2 - A'R = 0. \tag{39}$$

If we assume $A = au$, then equation (39) reduces to equation (35).

Thus, we have shown that the conditional Lie–Bäcklund symmetries obtained by Zhdanov can be derived by means of very particular invariant solutions of the heir-equations. We have also shown that more conditional Lie–Bäcklund symmetries than those found by Zhdanov can be obtained by means of other invariant solutions of the heir-equations.

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† In [21] there is a misprint which does not affect the given general solution.

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